MODEL OF AN ELASTIC MEDIUM OF SIMPLE STRUCTURE WITH THREE-DIMENSIONAL DISPERSION

(MODEL' UPRUGOI SREDY PROSTOI STRUKTURY S PROSTRANSTVENNOI DISPERSIEI)

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A number of papers (see, for example, [1-7] in which other references also are cited) have studied a generalized model of an elastic medium with an asymmetric stress tensor – the asymmetrical theory of elasticity. It is felt that the essential feature of this model is the inclusion of a characteristic parameter having the dimension of length. As a result, the model takes into account scale effects (micro-heterogeneity), and in this sense represents the next approximation as compared with the normal theory of elasticity*. At the same time, asymmetry of the stress tensor is evidently not a characteristic feature of the model and can be eliminated by appropriate determination of the stresses.

The aim of this study is to develop a general linear model of a macroscopically homogeneous elastic medium with a simple microstructure. The latter, in contrast to complex microstructures, is characterized by the fact that to define it kinematically it is sufficient to specify the field of displacement vectors.

As an initial micromodel we have taken the Born model of a simple lattice [9]. In section 1 we consider an algorithm which enables us to construct a one-to-one correspondence between the discrete structure and some analytical structure. We use this algorithm in section 2 to derive an exact continuous representation for the Born model which has the same form as in the normal theory of elasticity but with an operator form of Hooke's law.

The kernel of the corresponding integral operator can be expressed explicitly in terms of the force constants of the micromodel. In this case it is possible to derive a theory which retains the symmetry of the stress tensor and the usual expression for the density of strain energy. Therefore, it is no longer necessary to introduce a couple-stress tensor employed in the asymmetrical theory of elasticity.

From the phenomenological point of view the proposed model corresponds to the

* An analogous situation arises in the electrodynamics of a continuous medium with micro-heterogeneities, where the corresponding effect has been called (weak) three-dimensional dispersion [8].

general case of strong dispersion. In a zero order long-wave approximation we arrive at the normal theory of elasticity. In particular, it is shown that in contrast to the generally accepted approach [9 and 10] the correct symmetry of the tensor of the moduli of elasticity is a direct result of the requirement of invariance of energy with respect to rotation, independently of the presence of initial stresses.

In section 3 we consider the case of weak dispersion and a model of an isotropic medium with arbitrary dispersion. We establish the way in which the solution of an elasticity problem with dispersion can be obtained from the normal theory of elasticity by replacing the elastic constants by operators. We derive an expression for Green's tensor for an isotropic medium with strong or weak dispersion.

1. Consider a simple three-dimensional lattice defined by three non-coplanar vectors $e_\alpha (\alpha = 1, 2, 3)$. The lattice is invariant with respect to displacement along any of these vectors. As usual, we take as an elemental cell A, a parallelepiped constructed on e_α. It is convenient to introduce an oblique lattice system of coordinates x^α with the basis e_α and the metric tensor $g_{\alpha\beta} = e_\alpha \cdot e_\beta$. Then, the nodes of the lattice correspond to the vector $n = n^\alpha e_\alpha$, with integral components n^α.

Let $N(A)$ be a linear space of scalar or tensor functions $u(n)$ specified at the nodes of the lattice, and increasing as $|n| \to \infty$ not faster than some power of $|n|$. In the following we also should be able to interpolate the functions $u(n)$ by analytic functions $u(x)$ which are defined uniquely by certain natural conditions.

In addition to the interpolating function $u(x)$ we shall consider its Fourier transform, for which we retain the same notation but with argument k

$$u(k) = \int u(x)\, e^{ik\cdot x}\, dx \qquad (dx = dx^1 dx^2 dx^3) \tag{1.1}$$

In the k-space we introduce a system of coordinates k_β with the basis e^β dual to the e_α: $e^\beta \cdot e_\alpha = \delta_\alpha{}^\beta$, and we construct a parallelepiped $B\{-\pi \leqslant k_\beta \leqslant \pi\}$ with identified opposite points. Let $B(k)$ be a characteristic function of the region B, i.e. $B(k) = 1$ for $k \in B$ and $B(k) = 0$ for $k \overline{\in} B$. We denote the Fourier inverse transform $B(k)$, by $\delta_B(x)$

$$\delta_B(x) = \frac{1}{(2\pi)^3}\int B(k)\, e^{-ixk}\, dk = \frac{1}{\pi^3 v_A}\prod_{\alpha=1}^{3}\frac{\sin \pi (xe^\alpha)}{(xe^\alpha)} \tag{1.2}$$

where v_A is the volume of the cell A. Obviously, $\delta_B(0) = v_A{}^{-1}$ and $\delta_B(n) = 0$ at all the other nodes.

First let us suppose that $u(n)$ decreases sufficiently rapidly as $|n| \to \infty$ and let us define the function

$$u(k) = v_A B(k) \sum_n u(n)\, e^{ink} \tag{1.3}$$

Expression (1.3) is the expansion of $u(k)$ in the region B into a Fourier series with the coefficients $v_A\, u(n)$, and the truncating function $B(k)$ causes $u(k)$ to vanish outside B. For the Fourier inverse tranform $u(x)$ of the function $u(k)$, we find

$$u(x) = v_A \sum_n u(n)\, \delta_B(x - n) \tag{1.4}$$

It can be shown [11] that $u(x)$ continues analytically into the complex region as an entire function of the first order of growth and of the π-type.

Considering the properties of $\delta_B(x)$ we can easily see that $u(x)$, as defined by (1.4), is the required interpolating function. The one-to-one correspondence $u(n) \Rightarrow u(x) \to u(k)$ is ensured by the condition that the Fourier spectrum $u(k)$ of the function $u(x)$ is concentrated in the region B, and by the uniqueness of expansion (1.3). Note that in the one-dimensional case an analogous situation is encountered in information theory (the theorem of Kotel'nikov-Shannon [12]).

Consider now the general case of functions $u(n)$ which increase as $|n| \to \infty$ according to an exponential law. Let $K'(B)$ be a space of generalized functions $u(k)$ with supports concentrated in B, defined in the space K of basic infinitely differentiable functions [11]. We denote the Fourier inverse transform $K'(B)$, by $X'(A)$. It can be shown [11] that $u(x) \in X'(A)$ are regular functions, that the increase as $|x| \to \infty$ no more rapidly than the power of $|x|$ and continue analytically into the complex region as entire functions of the first order of growth and of the π-type. Identification of opposite points of the parallelepiped B is essential for eliminating (or, more precisely for the vanishing of) the functionals from $K'(B)$ the Fourier inverse transforms of which vanish at all nodes of the lattice. The series (1.3) now defines a generalized function $u(k) \in K'(B)$, the Fourier inverse transform $u(x)$ of which is in $X'(A)$ and assumes the value $u(n)$ at the nodes of the lattice. The proof of the latter assertion can be carried out by replacing the series (1.3) by an approximating trigonometrical polynomial and going to the limit. It can be shown that in $X'(A)$, $\delta_B(x)$ plays the part of the normal δ-function.

Thus we have established a specific isomorphism $N(A) \Rightarrow X'(A) \to K'(B)$. Each operation in $N(A)$ (not necessarily a linear operation) corresponds in fact to the operations in $X'(A)$ and $K'(B)$. In particular, suppose that $\Phi(n)$ is a function which decreases sufficiently rapidly as $|n| \to \infty$ (to be specific, we can consider that it is non-zero only at a finite number of points). Then

$$v_A \sum_n \Phi(n)\, u(n) = \int \Phi(x)\, u(x)\, dx = \left(\frac{1}{2\pi}\right)^3 \int \Phi(-k)\, u(k)\, dk$$
$$v_A \sum_{n'} \Phi(n-n')\, u(n') \Rightarrow \Phi(x) * u(x) \to \Phi(k)\, u(k) \tag{1.5}$$

Here * denotes integral convolution. Note that the multiplier $\Phi(k)$ can be arbitrarily defined outside the region B – the relations (1.5) are not invalidated. It will be found convenient in a number of cases to make use of this facility and continue $\Phi(k)$ analytically over the whole k-space.

2. As the basis of a macromodel of a continuous medium we propose the Born model of a simple lattice in a harmonic approximation [9]. This can readily be presented as follows. At the nodes of the lattice with an elemental cell A particles (atoms) of mass m are located. The interaction of the particles is realized by linear elastic constraints with given characteristics (atomic force constants) so that the elastic energy Φ is a quadratic function of the vector of particle displacement $u(n)$

$$\Phi = \sum_n u_\alpha(n)\, \Phi^\alpha(n) + \frac{1}{2} \sum_{nn'} u_\alpha(n)\, \Phi^{\alpha\beta}(n-n')\, u_\beta(n') \tag{2.1}$$

The force constants $\Phi^{\alpha}(n)$ take into account initial forces and in the absence of the latter, are equal to zero. The dependence of the force constants $\Phi^{\alpha\beta}(n - n')$ on the relative disposition of particles follows from the periodicity of the lattice. The requirement of invariance of energy with respect to translation and rotation impose definite conditions on the force constants [9]. We shall also assume that residual effects are limited (the case of non-ionic crystals), i.e. $\Phi^{\alpha}(n)$ and $\Phi^{\alpha\beta}(n)$ are non-zero only over a finite range of values of n.

Assuming, as usual, that the dependence on time is given by $\exp(-i\omega t)$, we can write the Lagrangian of the system in the form

$$2L = m\omega^2 g^{\alpha\beta} \sum_n \overline{u_\alpha(n)}\, u_\beta(n) - 2\sum_n \overline{u_\alpha(n)}\, \Phi^{\alpha}(n) - \\ - \sum_{nn'} \overline{u_\alpha(n)}\, \Phi^{\alpha\beta}(n - n')\, u_\beta(n') + 2\sum_n \overline{u_\alpha(n)}\, f^{\alpha}(n) \tag{2.2}$$

where the last term takes into account the contribution of external forces $f^{\alpha}(n)$.

Using the algorithm proposed above, we impose on the lattice an analytic structure, setting

$$u_\alpha(n) \Rightarrow u_\alpha(x), \qquad f^{\alpha}(n) \Rightarrow v_A\, q^{\alpha}(x), \qquad m = v_A\, \rho \tag{2.3}$$

Here $q^{\alpha}(x)$ has the meaning of density of the body forces and ρ is the mass density. At the same time it is convenient to replace $\Phi^{\alpha}(n)$ and $\Phi^{\alpha\beta}(n)$ by generalized functions the Fourier transforms of which extend analytically over the whole k-space

$$\Phi^{\alpha}(n) \to v_A \Psi^{\alpha}(k) = v_A \sum_n \Phi^{\alpha}(n)\, e^{in\cdot k}$$
$$\Phi^{\alpha\beta}(n) \to v_A{}^2 \Psi^{\alpha\beta}(k) = v_A \sum_n \Phi^{\alpha\beta}(n)\, e^{in\cdot k} \tag{2.4}$$

Introducing the notations

$$\langle u \mid v\rangle = \int \overline{u(x)}\, v(x)\, dx = \frac{1}{(2\pi)^3}\int \overline{u(k)}\, v(k)\, dk = \overline{\langle v \mid u\rangle}$$
$$\langle u \mid \Phi \mid v\rangle = \iint \overline{u(x)}\, \Phi(x - x')\, v(x')\, dx\, dx' = \frac{1}{(2\pi)^3}\int \overline{u(k)}\, \Phi(k)\, v(k)\, dk \tag{2.5}$$

and making use of (1.5), we can write the Lagrangian in the form

$$2L = \langle u_\alpha \mid \rho\omega^2 g^{\alpha\beta} \mid u_\beta\rangle - 2\langle u_\alpha \mid \Psi^{\alpha}\rangle - \langle u_\alpha \mid \Psi^{\alpha\beta} \mid u_\beta\rangle + 2\langle u_\alpha q^{\alpha}\rangle \tag{2.6}$$

Consider the structure $\Psi^{\alpha\beta}(k)$ and $\Psi^{\alpha}(k)$. Obviously the tensor $\Psi^{\alpha\beta}(k)$ is Hermitian *

$$\Psi^{\alpha\beta}(k) = \overline{\Psi^{\beta\alpha}}(k) = \Psi^{\beta\alpha}(-k) \tag{2.7}$$

We require the elastic energy

$$\Phi(u_\alpha) = \langle u_\alpha \mid \Psi^{\alpha}\rangle + {}^1/_2 \langle u_\alpha \mid \Psi^{\alpha\beta} \mid u_\beta\rangle \tag{2.8}$$

to be invariant with respect to translation and rotation. Suppose that the corresponding displacement is $u_\alpha{}^*$. Then it follows, that $\Phi(u_\alpha + u_\alpha{}^*) = \Phi(u_\alpha)$ for any u_α

(footnote on the next page)

Thus

$$\langle u_\alpha^* \mid \Psi^\alpha \rangle = 0, \quad \mathrm{Re}\,\langle u_\alpha^* \mid \Psi^{\alpha\beta} \mid u_\beta \rangle = 0, \quad \langle u_\alpha^* \mid \Psi^{\alpha\beta} \mid u_\beta^* \rangle = 0 \qquad (2.9)$$

Also, in the case of translation through the vector a_α the displacement $u_\alpha^*(k) \sim a_\alpha \delta(k)$, and for an infinitesimal rotation defined by the anti-symmetric tensor $a_{\alpha\beta}$, the displacement $u_\alpha^*(k) \sim a_{\alpha\beta}\partial^\beta \delta(k)$ $(\partial^\beta = \partial / \partial k_\beta)$.

From the first two conditions of (2.9) for the case of translation we find (here and in what follows the zero suffix detones the value of the function for $k = 0$)

$$\Psi_0{}^\alpha = 0, \qquad \Psi_0{}^{\alpha\beta} = 0 \qquad (2.10)$$

From this, in particular, we have that $\Psi^\alpha(k) = -ik_\nu \psi^{\nu\alpha}(k)$.

The same conditions for rotation taking into account (2.7), yield

$$\psi_0{}^{\nu\alpha} = \psi_0{}^{\alpha\nu}, \qquad [\partial^\mu \Psi^{\alpha\beta}(k)]_0 = 0 \qquad (2.11)$$

Thus $\Psi^{\alpha\beta}(k)$ can be expressed in the form

$$\Psi^{\alpha\beta}(k) = \psi^{\alpha\beta\nu\mu}(k)\, k_\nu k_\mu \qquad (2.12)$$

where the tensor $\psi^{\alpha\beta\nu\mu}(k)$, which is symmetric in $\nu\mu$ and Hermitian in $\alpha\beta$, is obviously uniquely defined by $\Psi^{\alpha\beta}(k)$.

In considering the third condition of (2.9) we note first of all, that it is not possible to substitute directly the values u_α^* into it, since the product of δ-functions has no meaning. This is associated with the fact that the non-linear functional $\Phi(u_\alpha)$ given in the form (2.1) or (2.8) is not, in general, defined for displacements which in the x-space are constant or vary linearly. To complete its definition it is necessary to construct the appropriate regularization**. Here we shall be guided by the following considerations. It is well-known that the density of the Lagrangian can be determined only to the accuracy of divergent terms. However the true energy density must be invariant with respect to rigid-body rotation. This condition determines the regularization uniquely. We introduce the tensor

$$c^{\nu\alpha\mu\beta} = S^{\nu\alpha\mu\beta}_{\varkappa\sigma\rho\tau}\psi^{\varkappa\sigma\rho\tau} = \psi^{\alpha\beta\nu\mu} + \psi^{\nu\beta\alpha\mu} - \psi^{\mu\beta\alpha\nu} \qquad (2.13)$$

Here S is a symmetrization operator, the components of which are readily expressed in terms of the Kronecher delta. It can easily be verified that in tensors symmetric in the second pair of indices (ψ satisfies this condition), the operator S has the inverse

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* *(footnote from previous page)*
In crystals of simple structure it is usual to assume invariance with respect to inversion, from which the symmetry of $\Phi^{\alpha\beta}(n)$ and $\Psi^{\alpha\beta}(k)$ in $\alpha\beta$ follows. However, in phenomenological models this restriction is unnecessary.

** It can be shown that in a discrete representation it defines a method of summation of divergent expressions.

$$\psi^{\alpha\beta\nu\mu} = {}^1/_2 (c^{\nu\alpha\mu\beta} + c^{\mu\alpha\nu\beta}) \tag{2.14}$$

and the tensor $c^{\nu\alpha\mu\beta}$ is symmetric in $\nu\alpha$. It follows from (2.14) that

$$\Psi^{\alpha\beta}(k) = \psi^{\alpha\beta\nu\mu}(k)\, k_\nu k_\mu = c^{\nu\alpha\mu\beta}(k)\, k_\nu k_\mu \tag{2.15}$$

Assuming

$$\zeta_{\nu\alpha}(k) = -ik_\nu u_\alpha(k), \quad \varepsilon_{\nu\alpha}(k) = \zeta_{(\nu\alpha)}(k), \quad \sigma^{\nu\alpha}(k) = c^{\nu\alpha\mu\beta}(k)\, \zeta_{\mu\beta}(k) \tag{2.16}$$

or in direct notations in the x-representation

$$\zeta = \operatorname{grad} u, \qquad \varepsilon = \operatorname{def} u, \qquad \sigma = \int c(x - x')\, \zeta(x')\, dx' \tag{2.17}$$

we can write the expression for Φ in the form

$$\Phi = \langle \zeta_{\nu\alpha} \,|\, \psi^{\nu\alpha} \rangle + {}^1/_2 \langle \varepsilon_{\nu\alpha} \,|\, \sigma^{\nu\alpha} \rangle \tag{2.18}$$

with the obvious order of operations. In this way the required regularization is determined.

For Φ to be invariant with respect to rotation it is now necessary and sufficient to require that

$$c_0^{\nu\alpha\,[\mu\beta]} = 0 \tag{2.19}$$

It can be shown that this is equivalent to the conditions

$$c_0^{\nu\alpha\mu\beta} = c_0^{\nu\alpha\beta\mu} = c_0^{\mu\beta\nu\alpha} \tag{2.20}$$

The same symmetry is displayed by $\psi_0^{\alpha\beta\nu\mu}$. The corresponding conditions must also be satisfied by the initial force constants.

For the coefficients of the expansion of the analytic function

$$c^{\nu\alpha\mu\beta}(k) = \sum_{p=0}^{\infty} c_p^{\nu\alpha\mu\beta\lambda_1\ldots\lambda_p} (-ik_{\lambda_1}) \cdots (-ik_{\lambda_p}) \tag{2.21}$$

we find, taking into account (2.4), (2.12) and (2.13),

$$c_p^{\nu\alpha\mu\beta\lambda_1\ldots\lambda_p} = \frac{1}{v_A} \frac{(-1)^{p+1}}{(p+2)!} S^{\nu\alpha\mu\beta}_{\varkappa\sigma\rho\tau} \sum_n \Phi^{\varkappa\sigma}(n)\, n^\rho n^\tau n^{\lambda_1} \ldots n^{\lambda_p} \tag{2.22}$$

In the zero order long-wave approximation $c_0^{\nu\alpha\mu\beta}$ plays the part of the usual tensor of elasticity moduli. The expression obtained for this quantity coincides with the well-known form [9].

From the foregoing it follows that the appropriate symmetry of $c_0^{\nu\alpha\mu\beta}$ follows exclusively from the requirement of invariance of energy with respect to translation and rotation. This result differs from the generally accepted concepts, based on [10] in, that, to obtain the correct symmetry of $c_0^{\nu\alpha\mu\beta}$, it is necessary to impose the condition of absence of initial stresses. The difference is associated with the fact that invariance of energy with respect to rotation, according to [10] leads to a relation between $\Phi^\alpha(n)$ and

$\Phi^{\alpha\beta}(n)$ (or between ${\psi_0}^{\nu\alpha}$ and ${\psi_0}^{\alpha\beta\nu\mu}$ in terms of the given work). However, the derivation of the corresponding formula in [10] is incorrect – the requirement of linear superposition, which is valid in a harmonic approximation, is not satisfied. Thus, by taking into account the independence of $\psi^{\nu\alpha}(k)$ and $\psi^{\alpha\beta\nu\mu}(k)$ we can consider without any loss of generality that initial stresses are absent, and we can discard the corresponding terms in (2.6) and (2.18), which we propose to do in what follows.

It can easily be shown that the real (imaginary) part of $\psi^{\alpha\beta\nu\mu}(k)$ is symmetric (antisymmetric) in $\alpha\beta$, symmetric in $\nu\mu$ and is an even (odd) function of k. Similarly, the real (imaginary) part of $c^{\nu\alpha\mu\beta}(k)$ is an even (odd) function of k. It follows that the imaginary parts of $\psi^{\alpha\beta\nu\mu}(k)$ and $c^{\nu\alpha\mu\beta}(k)$ are equal to zero for an isotropic medium and for a medium with central symmetry.

From the Lagrangian (2.6) we find the equations of motion which, when (2.15) is taken into account, assume the following form in the (k, ω) – and (x, t) – representations:

$$-\rho\omega^2 g^{\alpha\beta} u_\beta(k) + c^{\nu\alpha\mu\beta}(k)\, k_\nu k_\mu u_\beta(k) = q^\alpha(k) \tag{2.23}$$

$$\rho g^{\alpha\beta} u_\beta''(x) - \int c^{\nu\alpha\mu\beta}(x - x')\, \partial_\nu \partial_\mu u_\beta(x')\, dx' = q^\alpha(x) \tag{2.24}$$

If in (2.23) we put the body forces q^α equal to zero we can obtain a dispersion equation in the usual way which relates ω and k, and we can find the propagation velocity of waves which, in general, depend on k (three-dimensional dispersion). The requirement of stability, i.e. that $\omega^2(k)$ should not be negative for $k \in B$, imposes certain conditions concerning the type of the inequality, on $c^{\nu\alpha\mu\beta}(k)$.

Expression (2.18) for the elastic energy can be written in the form

$$\Phi = \int \varphi(x)\, dx, \qquad \varphi(x) = \frac{1}{2}\sigma^{\nu\alpha}(x)\, e_{\nu\alpha}(x) \tag{2.25}$$

The invariance of $\sigma^{\nu\alpha}(x)$, and, therefore, of $\Phi(x)$ as well, with respect to an infinitesimal rotation follows from (2.16) and (2.19). Thus $\varphi(x)$ can be identified with the strain energy density.

Let us write (2.24) and the last of Equations (2.17) in the form

$$-\rho u'' + \operatorname{div}\sigma = -q, \qquad \sigma = C\zeta = C \operatorname{grad} u \tag{2.26}$$

where the operator C is defined by the kernel $c^{\nu\alpha\mu\beta}(x)$. Comparing (2.26) and (2.25) we conclude that the symmetric tensor $\sigma^{\nu\alpha}$ can be treated as a stress tensor and the relation $\sigma = C\zeta$ as an operator form of Hooke's law. In the zero order long-wave approximation they coincide with the usual expressions. In view of the symmetry of the stress tensor it is necessary to introduce some form of supplementary couple-stress tensors as it was done in the asymmetrical theory of elasticity [1 to 7].

It should be emphasized that the symmetry of the stress tensor and the possibility of expressing the energy density in the form (2.25) are direct results of the regularization introduced above. These conditions no longer hold in the case of other regularization.

The model described may be considered to be an exact (one-to-one) continuous representation of the initial discrete model. The kernel of the operator C is determined by the force constants according to (2.22). In a phenomenological approach the Lagrangian (2.6) and Equations (2.23) and (2.24) define a most general model of a macroscopically homogeneous elastic medium with a simple structure exibiting a (strong) three-dimensional dispersion. If the region of permissible values of k is bounded, then the phenomenological model is exactly equivalent to some discrete model whose force constants are given by (2.4).

3. We shall consider now some particular models and take as first, the case of weak dispersion. Transition to the longitudinal wave approximation, strictly speaking, means that a certain class of functions is admissible whose spectrum is concentrated in a region of characteristic dimension $\varkappa \ll 1$. In this case in (2.21) or, what amounts to the same thing, in the expansion of the finite function $c^{\nu\alpha\mu\beta}(x)$ in a series of multipoles we can stop at a finite number of terms (with an estimate of error). Then the zero approximation gives the usual theory of elasticity.

To obtain the next approximation we start by considering the case when there is no central symmetry and, consequently, when the coefficient c_1 in (2.21) differs, in general, from zero. The operator form of Hooke's law then becomes

$$\sigma^{\nu\alpha}(x) = c_0^{\nu\alpha\mu\beta}\varepsilon_{\mu\beta}(x) + c_1^{\nu\alpha\mu\beta\lambda}\partial_\lambda\zeta_{\mu\beta}(x) \tag{3.1}$$

and the equations of motion will be of the third order in three-dimensional derivatives.

In the case when central symmetry exists, $c_1 = 0$ and the next approximation obtained is the second one with the following form of Hooke's law

$$\sigma^{\nu\alpha}(x) = c_0^{\nu\alpha\mu\beta}\varepsilon_{\mu\beta}(x) + c_2^{\nu\alpha\mu\beta\lambda_1\lambda_2}\partial_{\lambda_1}\partial_{\lambda_2}\zeta_{\mu\beta}(x) \tag{3.2}$$

Equations (2.24) are of the fourth order.

Now we shall consider the case of an isotropic medium with arbitrary dispersion. It can be shown that, in a Cartesian system of co-ordinates, the general expression

$$c^{\nu\alpha\mu\beta}(k) = \lambda(k)\delta^{\nu\alpha}\delta^{\mu\beta} + \mu(k)(\delta^{\alpha\beta}\delta^{\nu\mu} + \delta^{\mu\alpha}\delta^{\nu\beta}) \tag{3.3}$$

holds, where $\lambda(k)$ and $\mu(k)$ are real functions of the scalar argument $k^2 = k_\alpha k^\alpha$. It follows that in this case ζ in Hooke's law can be replaced by ε. The equations of motion (2.23) and (2.24) assume the form

$$-\rho\omega^2 u^\alpha(k) + \mu(k)k^2u^\alpha(k) + [\lambda(k) + \mu(k)]k^\alpha k^\beta u_\beta(k) = q^\alpha(k) \tag{3.4}$$

$$\rho u'' - M\Delta u - (\Lambda + M)\,\mathrm{grad}\,\mathrm{div}\,u = q \tag{3.5}$$

Here Λ and M are scalar operators with kernels $\lambda(x)$ and $\mu(x)$.

From the conditions of stability it follows, that

$$3\lambda(k) + 2\mu(k) > 0, \qquad \mu(k) > 0 \tag{3.6}$$

in the permissible range of variation of k.

Zero approximation is followed by the second approximation

$$\lambda(k) = \lambda_0 - \lambda_2 k^2, \qquad \mu(k) = \mu_0 - \mu_2 k^2 \tag{3.7}$$

If the medium is discrete and isotropic up to and including the second approximation, then the constants of the expansion can be expressed in terms of the micro-parameters

$$\lambda_s = 3p_s - 4q_s, \qquad \mu_s = -2p_s + q_s \qquad (s = 0,2) \tag{3.8}$$

where ($n^2 = n_\alpha n^\alpha$)

$$p_s = \frac{1}{30 \cdot 6^s v_A} \sum_n n^{2+s} \Phi^\alpha_{\cdot\alpha}(n), \qquad q_s = \frac{1}{30 \cdot 6^s v_A} \sum_n n^s n_\alpha n_\beta \Phi^{\alpha\beta}(n) \tag{3.9}$$

from which, taking into account (3.6), we find in particular that $p_0,\ q_0 < 0$.

Returning to Equations (3.5), we note, that the operators Λ and M commute with space and time derivatives. This enables us to formulate the following: the solution to a problem in the theory of elasticity with three-dimensional dispersion can be obtained from the solution to the corresponding problem in the classical theory of elasticity by replacing the Lamé parameters λ_0, and μ_0 with the operators Λ and M, with appropriate interpretation of combinations of operators*. The latter in general reduces to finding one-dimensional Fourier inverse transforms of the combinations of $\lambda(k)$ and $\mu(k)$. Note that this concept can obviously be extended to anisotropic problems, but the deciphering of operators in this case presents considerable difficulties.

As an example let us construct Green's tensor for the displacements of an isotropic medium with three-dimensional dispersion (statics). From the familiar expression for Green's tensor in the classical theory of elasticity [14], we find

$$U_{\alpha\beta}(k) = \frac{1}{k^2 \mu(k)} \delta_{\alpha\beta} - \frac{1}{k^4} \left[\frac{1}{\mu(k)} - \frac{1}{\lambda(k) + 2\mu(k)} \right] k_\alpha k_\beta \tag{3.10}$$

From this we obtain the integral form

$$U_{\alpha\beta}(x) = \delta_{\alpha\beta} \int \frac{\tau(x')}{r(x - x')}\, dx' - \int \chi(x')\, \partial_\alpha \partial_\beta r(x - x')\, dx' \tag{3.11}$$

The kernels $\tau(x)$ and $\chi(x)$ are given by the Fourier transforms

$$\tau(k) = \frac{1}{4\pi\mu(k)}, \qquad \chi(k) = \frac{\lambda(k) + \mu(k)}{8\pi\mu(k)[\lambda(k) + 2\mu(k)]} \tag{3.12}$$

In specific problems it is more convenient to use the direct expression for $U_{\alpha\beta}(x)$ with the aid of the theorem of residues. Thus, as a second approximation on the assumption that $\mu_2 < 0,\ \lambda_2 + 2\mu_2 < 0$, we find, that

$$U_{\alpha\beta}(x) = \frac{1}{\mu_0 a} f\left(\frac{r}{a}\right) + \partial_\alpha \partial_\beta \left[\frac{b}{\lambda_0 + 2\mu_0} g\left(\frac{r}{b}\right) - \frac{a}{\mu_0} g\left(\frac{r}{a}\right) \right]$$

Here

$$f(r) = \frac{1 - e^{-r}}{4\pi r}, \qquad g(r) = \frac{r}{8\pi} + f(r) \qquad \left(a^2 = -\frac{\mu_2}{\mu_0}, \quad b^2 = -\frac{\lambda_2 + 2\mu_2}{\lambda_0 + 2\mu_0} \right) \tag{3.13}$$

* For comparison refer to the analogous situation in the residual theory of creep [13].

Thus already the second approximation eliminates the divergence of Green's tensor as $r \to 0$. An analogous situation arises in electrodynamics with higher derivatives [15]. Strictly speaking, however, the study of waves with lengths of the order of the characteristic dimensions a and b is only possible within the framework of a model of the medium with strong dispersion (the author is indebted to G.I. Barenblatt for discussing this point). In this connection elimination of divergence in the second approximation theory is of a formal nature and is of value only for phenomenological models.

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